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CONSTRUCTING LEFSCHETZ FIBRATIONS VIA CYCLIC GROUP ACTIONS I

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Abstract. This article is first of two part series studying the Lefschetz fibrations over $\mathbb{S}^2$ using the cyclic group actions. In this article, using various cyclic group actions on product symplectic 4-manifolds $\mathbb{T}^2 \times \mathbb{T}^2$ and $\Sigma_2 \times \Sigma_2$ and applying the resolutions of cyclic quotient singularities, we study the monodromies of genus one and genus two Lefschetz fibrations over $\mathbb{S}^2$. The second article in this series will be devoted to the study of the monodromies of genus three Lefschetz fibrations over $\mathbb{S}^2$ and some constructions of new Lefschetz fibrations using the rational blow-down surgery.

Keywords: Symplectic 4-manifold, Lefschetz fibration, mapping class group, cyclic quotient singularities, rational blowdown.

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1 Introduction

The Lefschetz fibrations play a very important role in the study of the geometry and topology of symplectic 4-manifolds. In his seminal work Donaldson shows that every closed symplectic 4-manifold admits a structure of Lefschetz pencil which can be blown up at its base points to obtain a Lefschetz fibration over $\mathbb{S}^2$ (Donaldson, 1999). Conversely, it was shown by Gompf that the total space of a genus g Lefschetz fibration admits a symplectic structure, provided that the homology class of a regular fiber is nonzero Gompf et al. (1999). There is a well-known one-to-one correspondence between the Lefschetz fibrations and the words in the mapping class group. Namely, if we have a Lefschetz fibration $f : X \to \mathbb{S}^2$, then there is a corresponding word in the mapping class group of a regular fiber which is a composition of right-handed Dehn twists about the vanishing cycles of Lefschetz fibration f and vice versa (cf. Donaldson (1999); Gompf et al. (1999)).

One interesting problem in the theory of Lefschetz fibrations is to understand the topological interpretation of various relations in the mapping class group. For example, the well-known lantern and daisy relations in the mapping class group corresponds to the rational blowdown surgeries Endo & Gurtas (2010); Endo et al. (2011). Another interesting problem is whether any Lefschetz fibration over $\mathbb{S}^2$ admits a section (cf. Smith (2001)). Some results and constructions in these directions were obtained in (Endo & Gurtas (2010); Endo et al. (2011); Akhmedov & Monden (2016)). In Akhmedov & Monden (2016) the first author and Monden constructed new families of Lefschetz fibrations over $\mathbb{S}^2$ by applying the sequence of daisy substitutions, and conjugations to the hyperelliptic words

$$\left(c_1c_2\cdots c_{2g-1}c_{2g}c_{2g+1}^2c_{2g}c_{2g-1}\cdots c_2c_1\right)^2 = 1,$$

$$\left(c_1c_2\cdots c_{2g}c_{2g+1}\right)^{2g+2} = 1,$$

$$\left(c_1c_2\cdots c_{2g}\right)^{2(2g+1)} = 1$$

in the mapping class group of the closed orientable surface of genus g for any $g \geq 3$ and studied the sections of these Lefschetz fibrations.

In this paper, we will adopt another method, originally studied in Matsumoto (1996), to construct and study the topology of Lefschetz fibrations. We will construct the families of Lefschetz fibrations over $\mathbb{S}^2$ using various finite order cyclic group actions on genus g surface Σ_g and extending these actions to the product manifolds $\Sigma_g \times \Sigma_g$ for $g \geq 1$. We will use these actions to study Lefschetz fibrations of genus one and genus two over $\mathbb{S}^2$. By investigating the types of the singular fibers arising from the resolutions of cyclic quotient singularities, we will determine the monodromy of each singular fiber, which can be perturbed into several Lefschetz type singular fibers, and hence ultimately determine the global monodromies of our Lefschetz fibrations. In the second sequel to this paper Akhmedov & Nur Saglam Kadriye (2018), we study genus three Lefschetz fibrations over $\mathbb{S}^2$ and some applications of these Lefschetz fibrations using rational blowdown surgery.

The organization of our paper is as follows. In Section 2, we introduce some preliminaries on finite order cyclic actions, and mapping class groups. In Section 3, we describe in details the cyclic quotient singularities, their resolutions, topological invariants associated to them and some important theorems about them, and set up terminology and notation, which we will use throughout this paper. In Sections 4, we outline a general construction of Lefschetz fibrations starting from the product manifolds $\Sigma_g \times \Sigma_g$ for $g \geq 1$ and derive a lemma to be used in the proofs. In Sections 5 and 6, we present our main results.

2 Preliminaries

2.1 Finite order cyclic group actions

In this subsection, we will consider the various actions of finite cyclic groups on closed Riemann surfaces. Some of these family of cyclic group actions were considered in Akhmedov & Park (2008), where the authors used them for a different purpose. More specifically, in Akhmedov & Park (2008), the graphs of the diffeomorphisms generating these actions in the product 4-manifolds $\Sigma_g \times \Sigma_g$, for $g \geq 1$, were used to construct new symplectic 4-manifolds on and near Bogomolov-Miyaoka-Yau line via the branched cover construction.

2.1.1 Order $g+1$ cyclic action

Let g be a positive integer and Σ_g be a closed genus g Riemann surface. We will consider Σ_g as two concentric spheres connected via $g+1$ tubes. We take an orientation-preserving self-diffeomorphism $\gamma : \Sigma_g \to \Sigma_g$ which is the rotation of this surface by the angle $\dfrac{2\pi}{g+1}$. The diffeomorphism γ has 4 fixed points (the axis of rotation goes through two points on each sphere) and has order $g+1$ (see Figure 1 for the cases of $g=1$ and $g=2$).

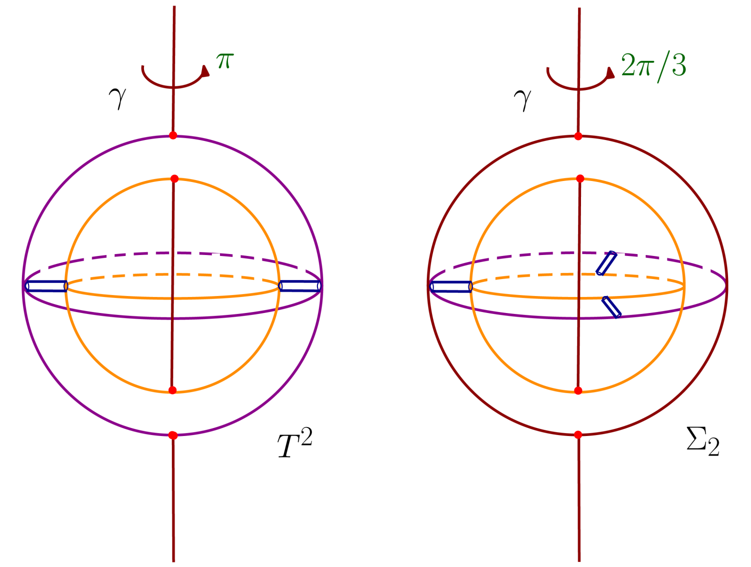

Figure 1: Order 2 action on $\mathbb{T}^2$ and order 3 action on Σ_2

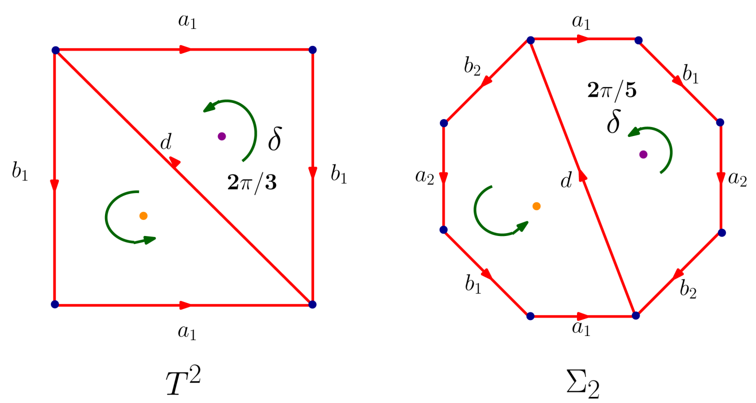

Figure 2: Order 3 action on $\mathbb{T}^2$ and order 5 action on Σ_2

2.1.2 Order $2g + 1$ cyclic action

Let g be a positive integer. We will think of the genus g surface Σ_g as a regular $4g$-gon with diametrically opposite edges identified so that the word given by the boundary of the $4g$-gon is

$$a_1 a_2 \cdots a_{2g} a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1}.$$

We cut this $4g$-gon into two $(2g+1)$-gons along a diagonal d such that the boundaries of the resulting two $(2g+1)$-gons are given by the words $a_1 a_2 \cdots a_{2g} d$ and $a_1^{-1} a_2^{-1} \cdots a_{2g}^{-1} d^{-1}$. Viewing each $(2g+1)$-gon as a regular polygon, let us rotate each $(2g+1)$-gon by the angle $\dfrac{2\pi}{2g+1}$, and then reglue them to obtain an orientation-preserving self-diffeomorphism $\delta : \Sigma_g \to \Sigma_g$ of order $2g + 1$ with 3 fixed points. (See Figure 2 for the cases of $g = 1$ and $g = 2$.)

2.1.3 Order g, $2g$, and $4g$ Actions

Let g be a positive integer. Let us think of the genus g surface Σ_g as a $4g$-gon with diametrically opposite edges identified so that the word given by the boundary of the $4g$-gon is

$$a_1 a_2 a_1^{-1} a_2^{-1} \cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1}.$$

By rotating this $4g$-gon by the angle $\dfrac{2\pi}{g}$, we obtain an orientation-preserving self-diffeomorphism $\alpha : \Sigma_g \to \Sigma_g$ of order g with 2 fixed points.

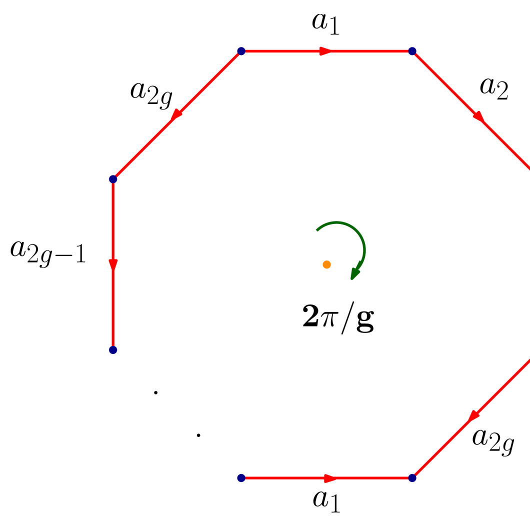

Figure 3: Order g action

Similarly, we can rotate this $4g$-gon by angles $2\pi/2g$ and $2\pi/4g$ to obtain orientation-preserving self-diffeomorphisms $\beta, \lambda : \Sigma_g \to \Sigma_g$ with 2 fixed points and of orders $2g$ and $4g$ respectively.

2.1.4 Composition with Hyperelliptic Action

Let us think of $\Sigma_g \subset \mathbb{R}^3$ such that the y-axis intersect it in $2g+2$ points and Σ_g is invariant under the $180°$ rotation around the y-axis (see Figure 4). This rotation defines a $\mathbb{Z}_2$-action $\tau_g : \Sigma_g \to \Sigma_g$ with $2g+2$ fixed points and is called *hyperelliptic involution*. By combining the

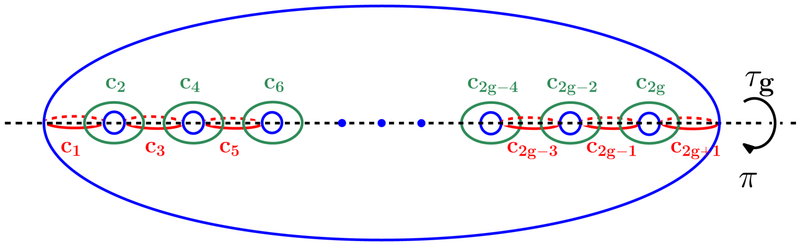

Figure 4: Hyperelliptic involution on Σ_g

hyperelliptic involution with the involutions that we have described above, we can obtain more finite order actions on Σ_g. Let us provide a few examples below.

- $\gamma \circ \tau_g$

  The hyperelliptic involution τ_2 defines a $\mathbb{Z}_2$ action on Σ_2 and γ defines a $\mathbb{Z}_3$ action on Σ_2. By combining these two actions, we obtain a $\mathbb{Z}_6$ action on Σ_2. More generally, the hyperelliptic involution τ_g defines a $\mathbb{Z}_2$ action on Σ_g and γ defines a $\mathbb{Z}_{g+1}$ action on Σ_g. By combining these two actions, we obtain a $\mathbb{Z}_{2(g+1)}$ action on Σ_g.

- $\delta \circ \tau_g$

  The hyperelliptic involution τ_2 defines a $\mathbb{Z}_2$ action on Σ_2 and δ defines a $\mathbb{Z}_5$ action on Σ_2. By combining them, we obtain a $\mathbb{Z}_{10}$ action on Σ_2. More generally, the hyperelliptic involution τ_g defines a $\mathbb{Z}_2$ action on Σ_g and δ gives a $\mathbb{Z}_{2g+1}$ action on Σ_g. By combining them, we obtain a $\mathbb{Z}_{2(2g+1)}$ action on Σ_g.

2.2 Mapping Class Group

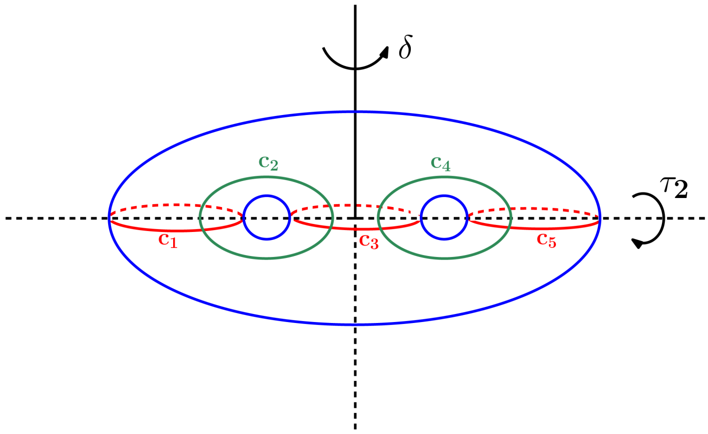

Figure 5: Hyperelliptic and vertical involution on Σ_2

The following lemma can be found in Luo (2000). For our purposes, we also state and prove a generalization of this lemma.

Lemma 1. *a) (Dehn, 2012) Let a and b be two simple loops in the torus $\Sigma_{1,0}$ so that they intersect transversely at one point. Let A and B be the positive Dehn-twist on a and b respectively. Then the standard symmetries of the torus are the following:*

the hyperelliptic involution $\tau_2 = ABABAB$, the 4-fold symmetry $\tau_4 = ABA$, and the 6-fold symmetry $\tau_6 = AB$.

b) (Birman, 1975) Let $a_1, \cdots, a_{r-1}$ be the pairwise disjoint arcs in the planar surface $\Sigma_{0,r}$, so that a_i joins the i-th boundary B_i to B_{i+1}. Let A_i be the half-twist about the arc a_i. Then $\tau_r = A_1 \cdots A_{r-1}$ and $\tau_{r-1} = A_1 \cdots A_{r-2}$ are $2\pi/r$ and $2\pi/(r-1)$-rotation of the surface sending a_i to a_{i+1} for $1 \leq i \leq r-3$.

c) (Birman, 1975) Let $C_1, \cdots, C_5$ be the positive Dehn-twists on the five simple loops $c_1, \cdots, c_5$ in the genus-2 surface (see Figure 5). Then the hyperelliptic involution $\tau_2 = C_1C_2C_3C_4C_5^2C_4C_3C_2C_1$ and the 5-fold symmetry is $\tau_5 = \tau_2C_1C_2C_3C_4$.

Let τ_g be the hyperelliptic involution obtained by rotating the genus g surface Σ_g along the horizontal axis by degree π (see Figure 4).

Lemma 2. *Let $C_1, \cdots, C_{2g+1}$ be the positive Dehn-twists along the $2g+1$ simple loops $c_1, ..., c_{2g+1}$ in the genus g surface (see Figure 4), where $g \geq 3$. Then the hyperelliptic involution τ_g on genus g surface Σ_g is $\tau_g = C_1C_2 \cdots C_{2g}C_{2g+1}^2C_{2g} \cdots C_2C_1$, and the $(2g+1)$-fold symmetry is $\tau_{2g+1} = \tau_gC_1C_2 \cdots C_{2g-1}C_{2g}$.*

Proof. The simple loops $C_1, \cdots, C_{2g+1}$ are invariant under the hyperelliptic involution τ_g. Therefore, the hyperelliptic involution τ_g is in the center of $\mathcal{M}_g$. Thus, there is a central extension

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{M}_{g,0} \xrightarrow{\rho} \mathcal{M}^*_{0,2g+2} \longrightarrow 1.$$

Let $f : \Sigma_{g,0} \to \Sigma_{g,0}$ be an orientation preserving homeomorphism. We can isotope f to an orientation preserving homeomorphism $\tilde{f}$ so that $\tilde{f} \circ \tau_g(x) = \tau_g \circ \tilde{f}(x)$ for all $x \in \Sigma_g$. Thus,

f induces a homeomorphism f_* on the quotient space $\Sigma_g/\tau_g = \Sigma_{0,2g+2}$ which is a sphere with $2g+2$ singular points. So, we can think of $[f_*]$ as an element in $\mathcal{M}^*_{0,2g+2}$. The map ρ sending $[f]$ to $[f_*]$ is a well-defined epimorphism with $Ker(\rho) = \langle \tau_g \rangle$ (cf. Birman (1975); Birman & Hilden (1973)).

The lifts of the relation $A_iA_{i+1}A_i = A_{i+1}A_iA_{i+1}$ gives the relation $\tau_g = C_1C_2\cdots C_{2g}C^2_{2g+1}C_{2g}\cdots C_2C_1$, which proves the first relation.

To show the second relation, observe by part b) of Lemma 1 that $A_1A_2\cdots A_{2g}$ is a period $(2g+1)$ element in $\mathcal{M}^*_{0,2g+2}$. It has two lifts in $\mathcal{M}_{2,0}$ given by $C_1C_2\cdots C_{2g}$ and $\tau_gC_1C_2\cdots C_{2g}$. The $2(2g+1)$-th power of both lifts are the identity. Namely,

$$(C_1C_2\cdots C_{2g})^{2(2g+1)}\,,$$

$$(\tau_gC_1C_2\cdots C_{2g})^{2(2g+1)}\,.$$

Next, we will prove that $\tau_gC_1C_2\cdots C_{2g}$ has order $2g+1$. Note that

$$H_1\left(\Sigma_{2,0}\right) = \langle [c_1], \cdots [c_{2g}]\rangle.$$

We can choose the orientation on c_i so that their algebraic intersections are $c_i \cdot c_{i+1} = 1$. The matrix representation of $c_1\cdots c_{2g}$ with respect to the basis $\{[c_1], \cdots, [c_{2g}]\}$ is given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ & & \cdots & & & \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -1 & 1 & -1 & 1 & \cdots & 1 \end{pmatrix}.$$

Note that $A^{2g+1} = -I$. The hyperelliptic involution τ_g induces the multiplication by -1 in homology. Therefore, $(\tau_gC_1C_2\cdots C_{2g})^{(2g+1)}$ induces identity on the first homology. Namely, $(\tau_gC_1C_2\cdots C_{2g})^{(2g+1)} = 1$. By Hurwitz theorem, the first homology detects the periodic homeomorphisms. $\Big((\tau_{2g+1})^{-1} \circ (\tau_gC_1C_2\cdots C_{2g})\Big)_*$ is trivial on the first homology group. Thus, $(\tau_{2g+1})^{-1} \circ (\tau_gC_1C_2\cdots C_{2g}) = 1$. Hence, we have $\tau_{2g+1} = \tau_gC_1C_2\cdots C_{2g}$.

□

3 Cyclic Quotient Singularities

Our presentation in this section follows Polizzi (2010) closely. We use the same terminology and notation as in Polizzi (2010). Let n and q be relatively prime natural numbers with $1 \leq q \leq n-1$, and let $\xi_n = e^{2\pi i/n}$ be a primitive n-th root of unity. Let us consider the order n action of the cyclic group $G = \mathbb{Z}_n = \langle \xi_n \rangle$ on $\mathbb{C}^2$ by diagonal matrices. By a slight normalization, we can assume that

$$G = \left\langle \begin{pmatrix} \xi_n & 0 \\ 0 & \xi_n^q \end{pmatrix} \right\rangle.$$

Then the quotient space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ contains a *cyclic quotient singularity of type* $\dfrac{1}{n}(1,q)$. Let q' denote the inverse of q in $(mod\ \ n)$. Namely, the unique integer $1 \leq q' \leq n-1$ such that $qq' \equiv 1\,(mod\ \ n)$. Then $X_{n_1,q_1} \cong X_{n,q}$ if and only if $n_1 = n$ and either $q_1 = q$ or $q_1 = q'$. The exceptional divisor $\mathrm{E} = \bigcup_{i=1}^k Z_i$ on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is an HJ-string

(Hirzebruch-Jung string). In other words, it is a configuration which consists of a collection of spheres $Z_1, \cdots, Z_k$ with self-intersection ≤ -2. They are ordered linearly so that $Z_i \cdot Z_{i+1} = 1$ for all i and $Z_i \cdot Z_j = 0$ if $|i - j| \geq 2$. More precisely, given the continued fraction expansion

$$\frac{n}{q} = [b_1, \cdots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_k}}}, \qquad b_i \geq 2,$$

the dual graph of E is as in Figure 6. Moreover,
$\dfrac{n}{q} = [b_1, \cdots, b_k]$ if and only if $\dfrac{n}{q'} = [b_k, \cdots, b_1]$.

Figure 6: Dual graph of E

Definition 1. *Let x be a cyclic quotient singularity of type $\frac{1}{n}(1, q)$ and let* E *be the corresponding HJ-string. If $\frac{n}{q} = [b_1, \cdots, b_k]$, we write* $\mathrm{E} : \frac{n}{q} = [b_1, \cdots, b_k]$ *and define*

$$\begin{aligned} l_x &= l(\mathrm{E}) = l\left(\frac{q}{n}\right) := k, \\ h_x &= h(\mathrm{E}) = h\left(\frac{q}{n}\right) := 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^{k}(b_i - 2), \\ e_x &= e(\mathrm{E}) = e\left(\frac{q}{n}\right) := k + 1 - \frac{1}{n}, \\ B_x &= B(\mathrm{E}) = B\left(\frac{q}{n}\right) := 2e_x - h_x = \frac{q + q'}{n} + \sum_{i=1}^{k} b_i. \end{aligned}$$

Definition 2. *A projective surface S is a* standard isotrivial fibration *if there a exists finite group G acting faithfully on two smooth projective curves C_1 and C_2 so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)/G$, where G acts diagonally on the product surface $C_1 \times C_2$. The two maps $\alpha_1 : S \to C_1/G$, $\alpha_2 : S \to C_2/G$ will be referred as the* natural projections*. If T is smooth, then $S = T$ is called* quasi-bundle*.*

It is well known that the stabilizer subgroup $H \subset G$ of a point $y \in C_2$ is a cyclic group ((Farkas & Kra, 1992) pg. 106). If the action of H on C_1 is free, then T is smooth along the fiber of $\sigma : T \to C_2/G$ over $\overline{y} \in C_2/G$, and this fiber consists of the curve C_1/H counted with multiplicity $|H|$. Thus, the smooth fibers of σ are all isomorphic to C_1. On the other hand, if the action of H on C_1 has a fixed point, say $x \in C_1$, then T has a cyclic quotient singularity over $\overline{(x, y)} \in T$.

The proof of the following theorem can be found in Polizzi (2010).

Theorem 1. *Let $\lambda : S \to T = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let us consider the natural projection $\alpha_2 : S \to C_2/G$. Take a point over $\overline{y} \in C_2/G$ and let F denote the schematic fiber of α_2 over $\overline{y}$. Then*

(i) The reduced structure of F is the union of an irreducible curve Y, called the central component of F, and either none or at least two mutually disjoint HJ-strings, each meeting Y

at one point, and each being contracted by λ to a singular point of T. These strings are in one-to-one correspondence with the branch points of $C_1 \to C_1/H$, where $H \subset G$ is the stabilizer of y.

(ii) The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.

(iii) Y is isomorphic to C_1/H, and has multiplicity equal to $|H|$ in F.

An analogous statement holds for the natural projection $\alpha_1 : S \to C_1/G$ as well.

The following proposition will be very useful for the computation purpose.

Proposition 1. *Let $\lambda : S \to T = (C_1 \times C_2)/G$ be a standard isotrivial fibration. Then the invariants of S are given by*

$$
\begin{aligned}
(i)\quad K_S^2 &= \frac{8\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} h_x;\\
(ii)\quad e(S) &= \frac{4\,(g(C_1)-1)\,(g(C_2)-1)}{|G|} + \sum_{x \in Sing(T)} e_x;\\
(iii)\quad q(S) &= g\,(C_1/G) + g\,(C_2/G)\,.
\end{aligned}
$$

Let us consider the minimal resolution of a cyclic quotient singularity $x \in T$. Let Y_1 and Y_2 be the strict transforms of C_1 and C_2. Then, by Theorem 1, we get a configuration as in Figure 7.

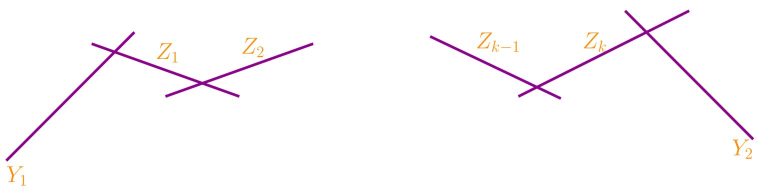

Figure 7: Resolution of a cyclic quotient singularity $x \in T$

Let F_1 and F_2 be the reducible fibers of $\alpha_2 : S \to C_2/G$ and $\alpha_1 : S \to C_1/G$, respectively. Then the curves Y_1 and Y_2 are the central components of F_1 and F_2 respectively and there exist $\lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_k \in \mathbb{N}$ such that

$$
\begin{aligned}
F_1 &= \rho_1 Y_1 + \sum_{i=1}^{k} \lambda_i Z_i + \Gamma_1,\\
F_2 &= \rho_2 Y_2 + \sum_{i=1}^{k} \mu_i Z_i + \Gamma_2,
\end{aligned}
$$

where the supports of both divisors Γ_1 and Γ_2 are union of HJ-strings disjoint from the Z_i. Moreover, if x is of type $\frac{1}{n}(1,q)$, then n divides both ρ_1 and ρ_2.

Definition 3. *We say that a reducible fiber F_1 of $\alpha_2 : S \to C_2/G$ is of* type $\left(\frac{q_1}{n_1}, \cdots, \frac{q_r}{n_r}\right)$ *if it contains exactly r HJ-strings $E_1, \cdots, E_r$, where each E_i is of type $\frac{1}{n_i}(1,q_i)$. The same definition holds for a reducible fiber F_2 of $\alpha_1 : S \to C_1/G$.*

Proposition 2. *(Polizzi, 2010) Let F_1 be of type $\left(\frac{q_1}{n_1}, \cdots, \frac{q_r}{n_r}\right)$ and let Y_1 be its central component. Then*

$$(Y_1)^2 = -\sum_{i=1}^{r} \frac{q_i}{n_i}.$$

Analogously, if F_2 is of type $\left(\frac{q_1}{n_1}, \cdots, \frac{q_r}{n_r}\right)$, then

$$(Y_2)^2 = -\sum_{i=1}^{r} \frac{q_i'}{n_i}.$$

4 Construction

In this section, we outline a general construction and compute some topological invariants associated with our construction and derive a lemma that will be used in our proofs.

Let us consider an order n self-diffeomorphism $\theta : \Sigma_g \to \Sigma_g$ with k fixed points. θ defines a cyclic group action of order n with k fixed points on Σ_g. We extend this action to the product 4-manifold $\Sigma_g \times \Sigma_g$ using $(\theta, \theta)(x, y)$. The quotient $S(g, n, k, t) = (\Sigma_g \times \Sigma_g)/\mathbb{Z}_n$ is a singular manifold with cyclic quotient singularities, where t denotes the type of the singular fibers. The singular 4-manifold $S(g, n, k, t)$ has nk singular points. By desingularizing these manifolds and by perturbing the singular fibers, we will obtain the families of Lefschetz fibrations $X(g, n, k, t)$ over $\mathbb{S}^2$. Since for the group actions that we consider in Section 2 the quotients $\Sigma_g/\mathbb{Z}_n$ are all spheres, the total spaces $X(g, n, k, t)$ of our families of Lefschetz fibrations are simply connected.

The desingularization process is done as follows. By first removing the neighborhoods of the singular points of $S(g, n, k, t)$, we get a manifold $S'(g, n, k, t)$ with $\partial S'(g, n, k, t) = \bigcup_1^{nk} L_{n_i, m_j}$. Next, we glue nk copies of C_{n_i, m_j} to $S'(g, n, k, t)$, where C_{n_i, m_j} is plumbing of certain disk bundles.

Lemma 3. *The total space $X(g, n, k, t)$ of the Lefschetz fibrations described above has Euler characteristic*

$$e\left(X(g, n, k, t)\right) = n \cdot e\left(F_s\right) + (2 - n) \cdot (2 - 2g),$$

where n denotes the order of the cyclic group action and F_s denotes the singular fiber of the Lefschetz fibration.

Proof. We can decompose the total space as

$$X(g, n, k, t) = \left(X(g, n, k, t) \setminus \bigcup_n F_s\right) \bigcup_n F_s.$$

$X(g, n, k, t) \setminus \bigcup_n F_s$ is a Σ_g bundle over $\mathbb{S}^2$ with n points removed (or analogously, D^2 with $n - 1$ points deleted). Let us denote the base by D^2_{n-1}. Then we have

$$\begin{aligned} e(X(g, n, k, t)) &= e\left(X(g, n, k, t) \setminus \bigcup_n F_s\right) + n \cdot e\left(F_s\right) \\ &= e\left(\Sigma_g\right) e\left(D^2_{n-1}\right) + n \cdot e\left(F_s\right) \\ &= (2 - 2g)(2 - n) + n \cdot e\left(F_s\right). \end{aligned}$$

$\square$

We will also make use of the following signature formula by Matsumoto and Endo in the computation of the signature of our Lefschetz fibrations.

Theorem 2. *(Endo, 2000), (Matsumoto, 1983, 1996) Let $f : X \to \mathbb{S}^2$ be a genus g hyperelliptic Lefschetz fibration. Let n and $s = \sum_{h=1}^{[g/2]} s_h$ be the numbers of the separating and non-separating vanishing cycles in the global monodromy of f, respectively, where s_h denotes the number of the vanishing cycles which separate the surface of genus g into two surfaces, one of which has genus h. Then, we have the following formula for the signature.*

$$\sigma(X) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h.$$

5 The genus one Lefschetz fibrations from finite order cyclic group actions on $\mathbb{T}^2$

In this section, we will study various finite order cyclic group actions on $\mathbb{T}^2$. By extending these actions to the product 4-manifolds $\mathbb{T}^2 \times \mathbb{T}^2$ and desingularizing the cyclic quotient singularities, we will construct genus one Lefschetz fibrations over $\mathbb{S}^2$. Each case will be presented in a subsection to make the material easy to follow.

5.1 $\mathbb{Z}_2$ Action On $\mathbb{T}^2$

5.1.1 Singular fiber of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

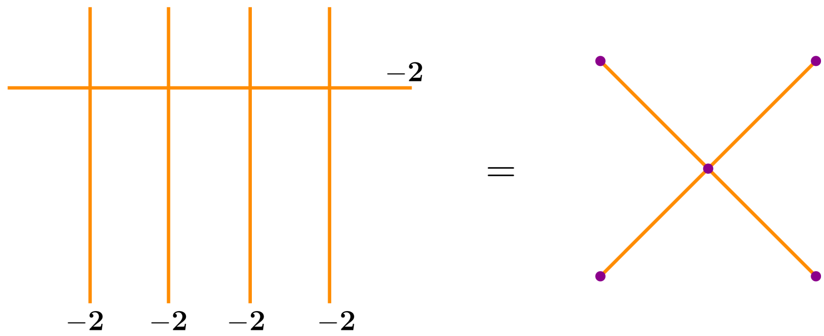

Figure 8: Order 2 action on $\mathbb{T}^2$ with 4 fixed points, singular fiber of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

$$(Y)^2 = -\sum_{i=1}^{4} \frac{1}{2} = -2$$

$$\frac{n_i}{q_i} = \frac{2}{1} = [2], \qquad 1 \le i \le 4$$

Theorem 3.

$$\begin{aligned} e(X(1,2,4,1)) &= 12, \qquad c_1^2(X(1,2,4,1)) = 0, \\ \sigma(X(1,2,4,1)) &= -8, \qquad \chi(X(1,2,4,1)) = 1. \end{aligned}$$

$X(1,2,4,1)$ is the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, and the global monodromy of the genus one Lefschetz fibration on $X(1,2,4,1)$ obtained via desingularization and then followed by perturbation is

$$((c_1 c_2)^3)^2 = 1.$$

Proof. It follows from Proposition 2 and consequently from Figure 8 that, there is no -1-sphere on the fiber and each singular fiber corresponds to type I_0^* in Table 1^* in Kirby & Melvin (1999) (see also the original classification given in Kodaira (1966)).

We compute the Euler characteristic of $X(1,2,4,1)$ using Lemma 3. Since each singular fiber has $e(F_s) = 5 \cdot 2 - 4 = 6$, we compute

$$e(X(1,2,4,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 2 \cdot 6 - 0(2-2) = 12.$$

Each singular fiber I_0^* has monodromy $(c_1c_2)^3 = (c_1c_2c_1)^2$.(cf. Kirby & Melvin (1999); Ogg (1966); Kodaira (1963)) Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,2,4,1)$ is

$$((c_1c_2c_1)^2)^2 = ((c_1c_2)^3)^2 = (c_1c_2)^6 = 1.$$

By Lemma 2, we get

$$\sigma(X(1,2,4,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 12 = -8.$$

Therefore, $c_1^2(X(1,2,4,1)) = 0$ and $\chi(X(1,2,4,1)) = 1$, which follows from the formulas $c_1^2(X) := 2e(X) + 3\sigma(X)$ and $\chi(X) := \dfrac{e(X)+\sigma(X)}{4}$. Using Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. $\square$

5.2 $\mathbb{Z}_3$ Action On $\mathbb{T}^2$

5.2.1 Singular fiber of type $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

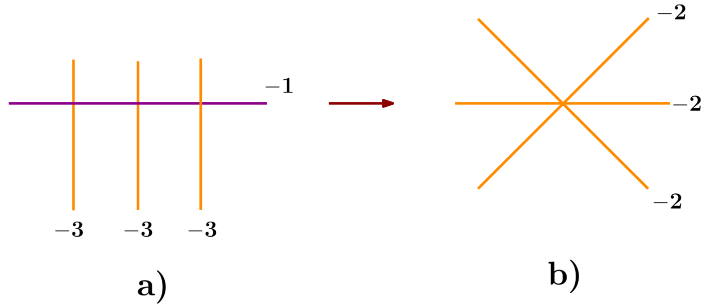

Figure 9: Order 3 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$(Y)^2 = -\sum_{i=1}^{3} \frac{1}{3} = -1,$$
$$\frac{n_i}{q_i} = \frac{3}{1} = [3], \qquad 1 \leq i \leq 3.$$

Theorem 4.

$$\begin{aligned} e(X(1,3,3,1)) &= 12, & c_1^2(X(1,3,3,1)) = 0, \\ \sigma(X(1,3,3,1)) &= -8, & \chi(X(1,3,3,1)) = 1. \end{aligned}$$

$X(1,3,3,1)$ is the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. The global monodromy of the corresponding genus one Lefschetz fibration on $X(1,3,3,1)$ obtained via desingularization and then followed by perturbation is

$$((c_1c_2)^2)^3 = 1.$$

Proof. Notice that the reducible fiber has a -1-sphere which is the central component and three -3-sphere intersecting it at three points as illustrated in Figure 9 a).

By blowing down -1-spheres, we obtain a manifold $X(1,3,3,1)$. After the blowdowns, each singular fiber consists of three -2-spheres intersecting at one point (See Figure 9 b)), which corresponds to type IV in Table 1 in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).

Each singular fiber has Euler characteristic $e(F_s) = 3 \cdot 2 - 2 = 4$. Hence,

$$e(X(1,3,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 3 \cdot 4 - (2-2) = 12.$$

Each singular fiber has monodromy $(c_1c_2)^2$. Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,3,3,1)$ is

$$((c_1c_2)^2)^3 = (c_1c_2)^6 = 1.$$

By Lemma 2, we get

$$\sigma(X(1,3,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 12 = -8.$$

Consequently, we have $c_1^2(X(1,3,3,1)) = 0$ and $\chi(X(1,3,3,1)) = 1$. Finally, by Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$. $\square$

5.2.2 Singular fiber of type $\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\right)$

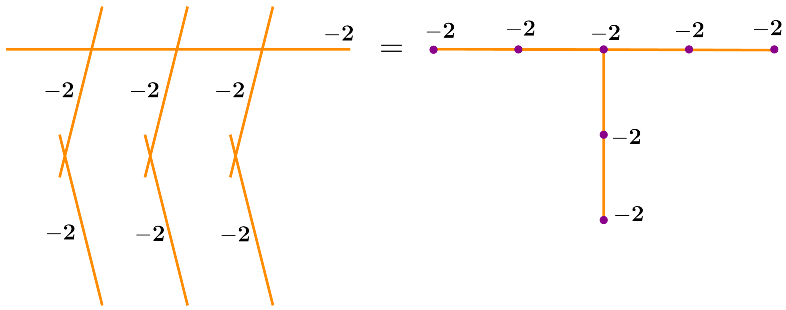

Figure 10: Order 3 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\dfrac{2}{3}, \dfrac{2}{3}, \dfrac{2}{3}\right)$

$$(Y)^2 = -\sum_{i=1}^{3} \frac{2}{3} = -2,$$

$$\frac{n_i}{q_i} = \frac{3}{2} = 2 - \frac{1}{2} = [2,2], \qquad 1 \leq i \leq 3.$$

Theorem 5.

$$\begin{aligned} e(X(1,3,3,2)) &= 24, \qquad c_1^2(X(1,3,3,2)) = 0, \\ \sigma(X(1,3,3,2)) &= -16, \qquad \chi(X(1,3,3,2)) = 2. \end{aligned}$$

$X(1,3,3,2)$ is the elliptic surface $E(2)$, and the global monodromy of the corresponding genus one Lefschetz fibration on $X(1,3,3,2)$ is

$$((c_1c_2)^4)^3 = 1.$$

Proof. In this case, we obtain a manifold $X(1,3,3,2)$ which has a singular fiber as given in Figure 10, which corresponds to type IV^* in Table 1* in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).
Each singular fiber has Euler characteristic $e(F_s) = 7 \cdot 2 - 6 = 8$. Hence,

$$e(X(1,3,3,2)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 3 \cdot 8 - (2-2) = 24.$$

Each singular fiber has monodromy $(c_1c_2)^4$. So, the global monodromy of the genus one Lefschetz fibration on $X(1,3,3,2)$ is

$$((c_1c_2)^4)^3 = (c_1c_2)^{12} = 1.$$

By Lemma 2, we get

$$\sigma(X(1,3,3,2)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 24 = -16.$$

Therefore, $c_1^2(X(1,3,3,2)) = 0$ and $\chi(X(1,3,3,2)) = 2$. Using Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(2)$.

□

5.3 $\mathbb{Z}_4$ Action On $\mathbb{T}^2$

5.3.1 Singular fiber of type $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$

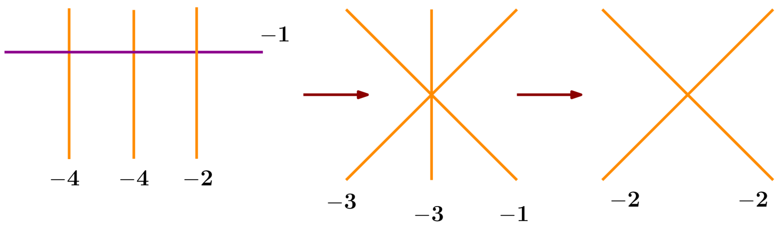

Figure 11: Order 4 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$

$$(Y)^2 = -\sum_{i=1}^{3} \frac{q_i}{n_i} = -1,$$

$$\frac{n_1}{q_1} = \frac{n_2}{q_2} = \frac{4}{1} = [4], \qquad \frac{n_3}{q_3} = \frac{2}{1} = [2].$$

Theorem 6.

$$\begin{aligned} e(X(1,4,3,1)) &= 12, \qquad c_1^2(X(1,4,3,1)) = 0, \\ \sigma(X(1,4,3,1)) &= -8, \qquad \chi(X(1,4,3,1)) = 1. \end{aligned}$$

$X(1,4,3,1)$ is the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, and the global monodromy of the genus one Lefschetz fibration on $X(1,4,3,1)$ is

$$(c_1c_2c_1)^4 = 1.$$

Proof. The singular fiber has a -1-sphere which is the central component and two -4-spheres and one -2-sphere each intersecting it at one point as shown in Figure 11 a).

Applying blow-down operation twice, we obtain a manifold $X(1,4,3,1)$ which has a singular fiber consists of two -2-spheres intersecting at one point (See Figure 11 b)), which corresponds to type III in Table 1 in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).

Each singular fiber has Euler characteristic $e(F_s) = 2 \cdot 2 - 1 = 3$. Hence,

$$e(X(1,4,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 4 \cdot 3 - 2 \cdot (2-2) = 12.$$

Each singular fiber has monodromy $c_1 c_2 c_1$. Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,4,3,1)$ is

$$(c_1 c_2 c_1)^4 = ((c_1 c_2)^3)^2 = (c_1 c_2)^6 = 1.$$

By Lemma 2, we get

$$\sigma(X(1,4,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 12 = -8.$$

Therefore, $c_1^2(X(1,4,3,1)) = 0$ and $\chi(X(1,4,3,1)) = 1$. Using Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.

□

5.3.2 Singular fiber of type $(\frac{1}{2}, \frac{3}{4}, \frac{3}{4})$

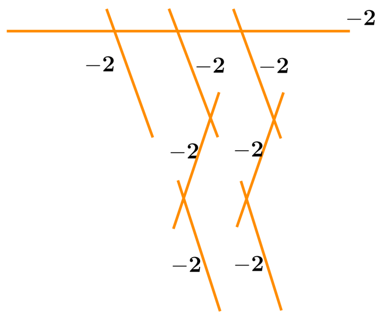

Figure 12: Order 4 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right)$

$$(Y)^2 = -\sum_{i=1}^{3} \frac{q_i}{n_i} = -2,$$

$$\frac{n_1}{q_1} = \frac{2}{1} = [2], \qquad \frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{4}{3} = [2,2,2].$$

Theorem 7.

$$\begin{aligned} e(X(1,4,3,2)) &= 36, & c_1^2(X(1,4,3,2)) = 0, \\ \sigma(X(1,4,3,2)) &= -24, & \chi(X(1,4,3,2)) = 3. \end{aligned}$$

$X(1,4,3,2)$ is the elliptic surface $E(3)$ and the global monodromy of the genus one Lefschetz fibration on $X(1,4,3,2)$ is $((c_1 c_2 c_1)^3)^4 = 1$.

Proof. In this case, we obtain a 4-manifold $X(1,4,3,2)$ which has a singular fiber as in Figure 12, which corresponds to type III^*, E_7 singularity, in Table 1^* in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).

Each singular fiber has Euler characteristic $e(F_s) = 8 \cdot 2 - 7 = 9$. Hence,

$$e(X(1,4,3,2)) = n \cdot e(F_s) + (2-n)\cdot(2-2g) = 4 \cdot 9 - 2 \cdot (2-2) = 36.$$

Each singular fiber has monodromy $(c_1c_2c_1)^{-1} = (c_1c_2c_1)^3$. Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,4,3,2)$ is $((c_1c_2c_1)^3)^4 = (c_1c_2c_1)^{12} = ((c_1c_2)^2)^{12} = (c_1c_2)^{24} = 1$.

By Lemma 2, we get

$$\sigma(X(1,4,3,2)) = -\frac{g+1}{2g+1}\cdot n - \sum_{h=1}^{[g/2]}\left(\frac{4h\,(g-h)}{2g+1} - 1\right)s_h = -\frac{2}{3}\cdot 36 = -24.$$

Therefore, $c_1^2(X(1,4,3,2)) = 0$ and $\chi(X(1,4,3,2)) = 3$. Using Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(3)$.

□

5.4 $\mathbb{Z}_6$ Action On $\mathbb{T}^2$

5.4.1 Singular fiber of type $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$

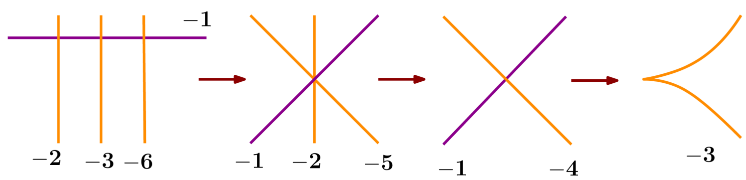

Figure 13: Order 6 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$

$$(Y)^2 = -\sum_{i=1}^{3}\frac{q_i}{n_i} = -1,$$

$$\frac{n_1}{q_1} = \frac{2}{1} = [2], \qquad \frac{n_2}{q_2} = \frac{3}{1} = [3], \qquad \frac{n_3}{q_3} = \frac{6}{1} = [6].$$

Theorem 8.

$$\begin{aligned} e(X(1,6,3,1)) &= 12, \qquad c_1^2(X(1,6,3,1)) = 0, \\ \sigma(X(1,6,3,1)) &= -8, \qquad \chi_h(X(1,6,3,1)) = 1. \end{aligned}$$

$X(1,6,3,1)$ is the elliptic surface $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$, and the global monodromy of the genus one Lefschetz fibration on $X(1,6,3,1)$ is

$$(c_1c_2)^6 = 1.$$

Proof. The reducible fiber is as illustrated in Figure 13.

Once we apply blow-down operation three times, we obtain a manifold $X(1,6,3,1)$ which has a singular fiber a cusp (See Figure 13), which corresponds to type II in Table 1 in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).

Each singular fiber has Euler characteristic $e(F_s) = 2$. Hence,

$$e(X(1,6,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 6 \cdot 2 - 4 \cdot (2-2) = 12.$$

Each singular fiber has monodromy c_1c_2. Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,6,3,1)$ is $(c_1c_2)^6 = 1$.

By Lemma 2, we get

$$\sigma(X(1,6,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 12 = -8.$$

Consequently, we have $c_1^2(X(1,6,3,1)) = 0$ and $\chi(X(1,6,3,1)) = 1$. By Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$.

$\square$

5.4.2 Singular fiber of type $(\frac{1}{2}, \frac{2}{3}, \frac{5}{6})$

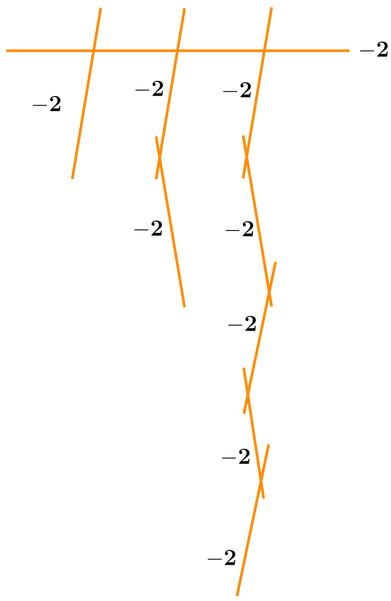

Figure 14: Order 6 action on $\mathbb{T}^2$ with 3 fixed points, singular fiber of type $\left(\frac{1}{2}, \frac{2}{3}, \frac{5}{6}\right)$

$$(Y)^2 = -\sum_{i=1}^{3} \frac{q_i}{n_i} = -2,$$

$$\frac{n_1}{q_1} = \frac{2}{1} = [2], \qquad \frac{n_2}{q_2} = \frac{3}{2} = [2,2], \qquad \frac{n_3}{q_3} = \frac{6}{5} = [2,2,2,2,2].$$

Theorem 9.

$$\begin{aligned} e(X(1,6,3,2)) &= 60, & c_1^2(X(1,6,3,2)) = 0, \\ \sigma(X(1,6,3,2)) &= -40, & \chi(X(1,6,3,2)) = 5. \end{aligned}$$

$X(1,6,3,2)$ is the elliptic surface $E(5)$, and the global monodromy of the genus one Lefschetz fibration on $X(1,6,3,2)$ is $((c_1c_2)^5)^6 = 1$.

Proof. In this case, we obtain a manifold $X(1,6,3,2)$ which has a singular fiber as in Figure 14, which corresponds to type II^*, E_8 singularity, in Table 1^* in Kirby & Melvin (1999) (Also see Ogg (1966); Kodaira (1963)).

Each singular fiber has Euler characteristic $e(F_s) = 9 \cdot 2 - 8 = 10$. Hence,

$$e(X(1,6,3,2)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 6 \cdot 10 - 4 \cdot (2-2) = 60.$$

Each singular fiber has monodromy $(c_1c_2)^5 = (c_1c_2)^{-1}$. Thus, the global monodromy of the genus one Lefschetz fibration on $X(1,6,3,2)$ is $((c_1c_2)^5)^6 = (c_1c_2)^{30} = 1$. By Lemma 2, we get

$$\sigma(X(1,6,3,2)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h = -\frac{2}{3} \cdot 60 = -40.$$

Consequently, we have $c_1^2(X(1,6,3,2)) = 0$ and $\chi(X(1,6,3,2)) = 5$. Using Kodaira's classification of elliptic fibrations, we conclude that the total space is $E(5)$.

□

6 The genus two Lefschetz fibrations from finite order cyclic group actions on Σ_2

Let G be a finite cyclic group acting faithfully on a closed Riemann surface Σ_2 of genus 2. Let us assume that $g(\Sigma_2/G) = 0$. We will consider the diagonal action of G on $\Sigma_2 \times \Sigma_2$. By desingularization of the cyclic quotient singularities of $(\Sigma_2 \times \Sigma_2)/G$ and perturbing the singular fibers, we will construct the genus two Lefschetz fibrations over $\mathbb{S}^2$.

6.1 $\mathbb{Z}_2$ Action On Σ_2

6.1.1 Singular fiber of type $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

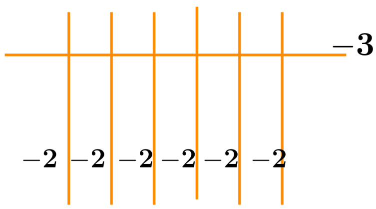

Figure 15: hyperelliptic action on Σ_2 with 6 fixed points, singular fiber of type $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$$n_i = 2 \qquad q_i = 1 \qquad i = 1, \cdots, 6,$$

$$(Y)^2 = -\sum_{i=1}^{6} \frac{q_i}{n_i} = -3,$$

$$\frac{n_i}{q_i} = \frac{2}{1} = [2] \qquad i = 1, \cdots, 6.$$

In this case, the singular fibers (See Figure 15) correspond to type I^*_{0-0-0} in Namikawa & Ueno (1973) (Also type 33 in the table on pg. 359 in Ogg (1966)).

Theorem 10.

$$\begin{aligned} e(X(2,2,6,1)) &= 16, \qquad c_1^2(X(2,2,6,1)) = -4, \\ \sigma(X(2,2,6,1)) &= -12, \qquad \chi(X(2,2,6,1)) = 1. \end{aligned}$$

$X(2,2,6,1)$ is diffeomorphic to $\mathbb{C}P^2\#13\overline{\mathbb{CP}}^2$, and the global monodromy of the genus two Lefschetz fibration on $X(2,2,6,1)$ obtained via desingularization and then followed by perturbation is

$$\left(c_1c_2c_3c_4c_5^2c_4c_3c_2c_1\right)^2 = 1.$$

Proof. Using the description of singular fibers given above, we compute that each singular fiber F_s has Euler characteristic

$$e(F_s) = 7\cdot 2 - 6 = 8.$$

By applying Lemma 3, we have

$$e(X(2,2,6,1)) = n\cdot e(F_s) + (2-n)\cdot(2-2g) = 2\cdot 8 - 0\cdot(2-4) = 16.$$

The monodromy of the singular fibers can be determined using Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). There are two singular fibers, and each has monodromy $c_1c_2c_3c_4c_5^2c_4c_3c_2c_1$.

Consequently, the global monodromy of the genus two Lefschetz fibration on $X(2,2,6,1)$ is $\left(c_1c_2c_3c_4c_5^2c_4c_3c_2c_1\right)^2 = 1$.

By Lemma 2, we have

$$\sigma(X(2,2,6,1)) = -\frac{g+1}{2g+1}\cdot n - \sum_{h=1}^{[g/2]}\left(\frac{4h\,(g-h)}{2g+1} - 1\right)s_h = -\frac{3}{5}\cdot 20 = -12$$

Therefore, $c_1^2(X(2,2,6,1)) = -4$ and $\chi(X(2,2,6,1)) = 1$.

Using the classification of genus two Lefschetz fibrations with non-separating singular fibers, which is due to Chakiris, (see Theorem 5.5 in Smith (1999)), we see that $X(2,2,6,1)$ is diffeomorphic to $\mathbb{C}P^2\#13\overline{\mathbb{CP}}^2$.

□

6.2 $\mathbb{Z}_3$ Action On Σ_2

6.2.1 Singular fiber of type $(\frac{1}{3},\frac{1}{3},\frac{2}{3},\frac{2}{3})$

$$n_i = 3 \quad i = 1,\cdots,4, \qquad q_1 = q_2 = 1, \;\; q_3 = q_4 = 2$$

$$(Y)^2 = -\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{2}{3}\right) = -2$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{3}{1} = [3], \\ \frac{n_3}{q_3} &= \frac{n_4}{q_4} = \frac{3}{2} = 2 - \frac{1}{2} = [2,2] \end{aligned}$$

In this case, the singular fibers (See Figure 16) correspond to type III in Namikawa & Ueno (1973) on pg. 155 (Also type 42 in the table on pg. 359 in Ogg (1966)).

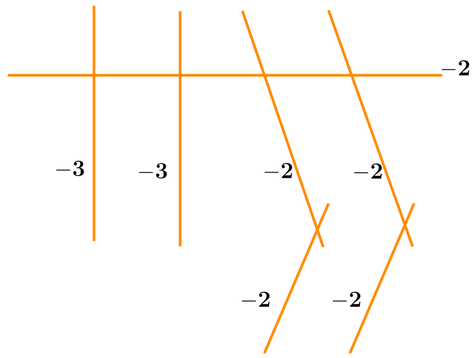

Figure 16: Order 3 action on Σ_2 with 4 fixed points, singular fiber of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Theorem 11.

$$\begin{aligned} e(X(2,3,4.1)) &= 26, \qquad c_1^2(X(2,3,4.1)) = -2, \\ \sigma(X(2,3,4.1)) &= -18, \qquad \chi(X(2,3,4.1)) = 2. \end{aligned}$$

$X(2,3,4.1)$ is diffeomorphic to $K3\#2\overline{\mathbb{CP}}^2$, and the global monodromy of the genus two Lefschetz fibration on $X(2,3,4.1)$ obtained via desingularization and then followed by perturbation is $(c_1c_2c_3c_4c_5)^6 = 1$.

Proof. Using the description of singular fibers, we see that each singular fiber F_s has Euler characteristic

$$e(F_s) = 7 \cdot 2 - 6 = 8.$$

Using Lemma 3, we have

$$e(X(2,3,4.1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 3 \cdot 8 + (-1)(2-4) = 26.$$

There are 3 singular fibers each has monodromy given by $(c_1c_2c_3c_4c_5)^2$. To determine the monodromy of the singular fibers, we make use of Ishizaka's classification of the periodic monodromies given in Ishizaka (2007).

Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,3,4.1)$ is $((c_1c_2c_3c_4c_5)^2)^3 = (c_1c_2c_3c_4c_5)^6 = 1$.

By Lemma 2, we get

$$\sigma(X(2,3,4.1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 30 = -18.$$

Consequently, $c_1^2(X(2,3,4.1)) = -2$ and $\chi(X(2,3,4.1)) = 2$.

Using the classification of genus two Lefschetz fibrations with non-separating singular fibers Smith (1999), we see that $X(2,3,4,1)$ is diffeomorphic to $K3\#2\overline{\mathbb{CP}}^2$.

□

6.3 $\mathbb{Z}_4$ Action On Σ_2

6.3.1 Singular fiber of type $(\frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{3}{4})$

$$n_i = 4 \quad i = 1, \cdots, 4, \qquad q_1 = 1, \quad q_2 = q_3 = 2, \quad q_4 = 3,$$

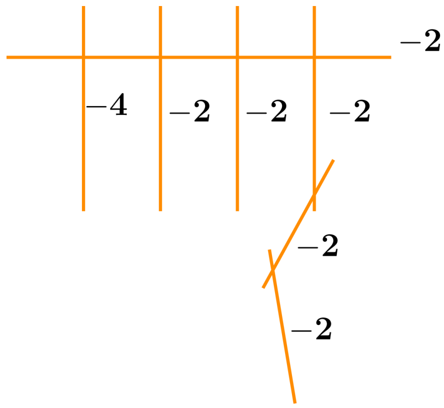

Figure 17: Order 4 action on Σ_2 with 4 fixed points, singular fiber of type $\left(\frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{3}{4}\right)$

$$(Y)^2 = -\left(\frac{1}{4} + \frac{2}{4} + \frac{2}{4} + \frac{3}{4}\right) = -2,$$

$$\begin{aligned}\frac{n_1}{q_1} &= \frac{4}{1} = [4],\\ \frac{n_2}{q_2} &= \frac{n_3}{q_3} = \frac{4}{2} = [2],\\ \frac{n_4}{q_4} &= \frac{4}{3} = [2,2,2]\end{aligned}$$

In this case, the singular fibers correspond to type VI in Namikawa & Ueno (1973) on pg. 156 (Also type 4 in the table on pg. 357 in Ogg (1966)).

Theorem 12.

$$\begin{aligned}e(X(2,4,4,1)) &= 36, \qquad c_1^2(X(2,4,4,1)) = 0,\\ \sigma(X(2,4,4,1)) &= -24, \qquad \chi(X(2,4,4,1)) = 3.\end{aligned}$$

$X(2,4,4,1)$ is the elliptic surface $E(3)$, and the global monodromy of the genus two Lefschetz fibration on $X(2,4,4,1)$ is

$$(c_1c_2c_3c_4c_5^2c_4c_3c_2c_2)^4 = 1.$$

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 7 \cdot 2 - 6 = 8.$$

Hence, again using Lemma 3, we get

$$e(X(2,4,4,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 4 \cdot 8 + (2-4)(2-4) = 36.$$

There are 4 singular fibers each has monodromy $c_1c_2c_3c_4c_5^2c_4c_3c_2c_2$. We determine the above monodromy using Ishizaka's classification of the periodic monodromies given in Ishizaka (2007).

Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,4,4,1)$ is $(c_1c_2c_3c_4c_5^2c_4c_3c_2c_2)^4 = 1$.
By Lemma 2, we have

$$\sigma(X(2,4,4,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 40 = -24.$$

Consequently, $c_1^2(X(2,4,4,1)) = 0$ and $\chi(X(2,4,4,1)) = 3$.

It follows by the classification result in Smith (1999) for the genus two Lefschetz fibrations that $X(2,4,4,1)$ is diffeomorphic to the elliptic surface $E(3)$. □

6.4 $\mathbb{Z}_5$ Action On Σ_2

6.4.1 Singular fiber of type $(\frac{1}{5}, \frac{1}{5}, \frac{3}{5})$

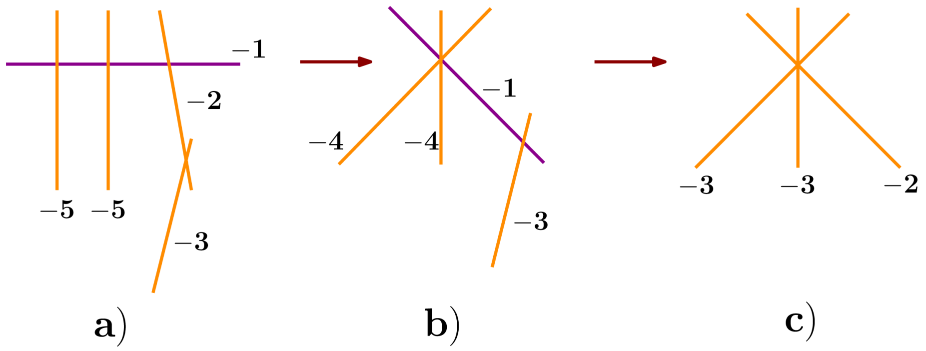

Figure 18: Order 5 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)$

$$n_i = 5 \quad 1 \leq i \leq 3, \qquad q_1 = q_2 = 1, \quad q_3 = 3$$
$$(Y)^2 = -\left(\frac{1}{5} + \frac{1}{5} + \frac{3}{5}\right) = -1$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{5}{1} = [5], \\ \frac{n_3}{q_3} &= \frac{5}{3} = 2 - \frac{1}{3} = [2,3] \end{aligned}$$

In this case, the singular fibers has a central -1-sphere as illustrated in Figure 18 a).

Now, we blow down the central -1-sphere and get a manifold which has the singular fiber with the configuration as in Figure 18 b). The new singular fiber still has a central -1-sphere. Blowing down once more, we get a singular fiber as in Figure 18c which corresponds to type 36 in the table on pg. 359 in Ogg (1966)(see also type $IX-2$ in Namikawa & Ueno (1973) on pg. 157).

Theorem 13.

$$\begin{aligned} e(X(2,5,3,1)) &= 26, \qquad c_1^2(X(2,5,3,1)) = -2, \\ \sigma(X(2,5,3,1)) &= -18, \qquad \chi(X(2,5,3,1)) = 2. \end{aligned}$$

$X(2,5,3,1)$ is diffeomorphic to $K3\#2\overline{\mathbb{CP}}^2$, and the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,1)$ is $(c_1c_2c_3c_4c_5^2)^5 = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 3 \cdot 2 - 2 = 4.$$

Hence,

$$e(X(2,5,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 5 \cdot 4 + (-3) \cdot (2-4) = 26.$$

There are 5 singular fibers, and each has monodromy $c_1c_2c_3c_4c_5^2$. The later follows from Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,1)$ is $(c_1c_2c_3c_4c_5^2)^5 = 1$.

Next, by Endo's signature formula for hyperelliptic Lefschetz fibrations, we compute

$$\sigma(X(2,5,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 30 = -18.$$

Consequently, we have $c_1^2(X(2,5,3,1)) = -2$ and $\chi(X(2,5,3,1)) = 2$.

Finally, using the classification of genus two Lefschetz fibrations with non-separating singular fibers Smith (1999), we see that $X(2,5,3,1)$ is diffeomorphic to $K3\#2\overline{\mathbb{CP}}^2$.

□

6.4.2 Singular fiber of type $(\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$

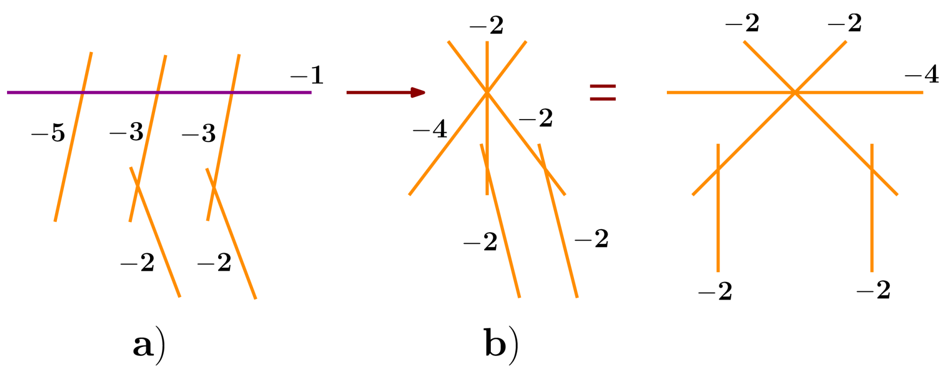

Figure 19: Order 5 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

$$n_i = 5 \quad 1 \leq i \leq 3, \qquad q_1 = 1, \quad q_2 = q_3 = 2$$

$$(Y)^2 = -\left(\frac{1}{5} + \frac{2}{5} + \frac{2}{5}\right) = -1$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{5}{1} = [5], \\ \frac{n_2}{q_2} &= \frac{n_3}{q_3} = \frac{5}{2} = 3 - \frac{1}{2} = [3,2] \end{aligned}$$

In this case the singular fibers has a central -1-sphere as illustrated in Figure 19a.

Now, we blow down the central -1-sphere and get $X(2,5,3,2)$ which has a singular fiber with the configuration as in Figure 19b. The new fiber corresponds to type 8 in the table on pg. 357 in Ogg (1966) (see also Namikawa & Ueno (1973), type $IX-1$ on pg. 157).

Theorem 14.

$$\begin{aligned} e(X(2,5,3,2)) &= 36, \qquad c_1^2(X(2,5,3,2)) = 0, \\ \sigma(X(2,5,3,2)) &= -24, \qquad \chi(X(2,5,3,2)) = 3. \end{aligned}$$

$X(2,5,3,2)$ is the Horikawa surface (see Fuller (2003); Akhmedov & Monden (2016)), and the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,2)$ is

$$(c_1c_2c_3c_4)^{10} = 1.$$

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 5 \cdot 2 - 4 = 6.$$

Hence,

$$e(X(2,5,3,2)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 5 \cdot 6 + (-3) \cdot (2-4) = 36.$$

There are 5 singular fibers each has monodromy $(c_1c_2c_3c_4)^2$. We determine the above monodromy using Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,2)$ is

$$\left((c_1c_2c_3c_4)^2\right)^5 = (c_1c_2c_3c_4)^{10} = 1.$$

By Endo's signature formula for hyperelliptic Lefschetz fibrations,

$$\sigma(X(2,5,3,2)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 40 = -24.$$

Therefore, $c_1^2(X(2,5,3,2)) = 0$ and $\chi(X(2,5,3,2)) = 3$.
$X(2,5,3,2)$ is one of the Horikawa's surface with its genus two fibration (Smith (1999)). Its total space is not non-minimal.

□

6.4.3 Singular fiber of type $(\frac{2}{5}, \frac{4}{5}, \frac{4}{5})$

$$n_i = 5 \quad 1 \le i \le 3, \qquad q_1 = 2, \;\; q_2 = q_3 = 4$$

$$(Y)^2 = -\left(\frac{2}{5} + \frac{4}{5} + \frac{4}{5}\right) = -2$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{5}{2} = [3,2], \\ \frac{n_2}{q_2} &= \frac{n_3}{q_3} = \frac{5}{4} = [2,2,2,2]. \end{aligned}$$

In this case, as can be seen in Figure 20, the singular fibers corresponds to type 21 in the table on pg. 358 in Ogg (1966) (see also Namikawa & Ueno (1973) type $IX-3$ on pg.157).

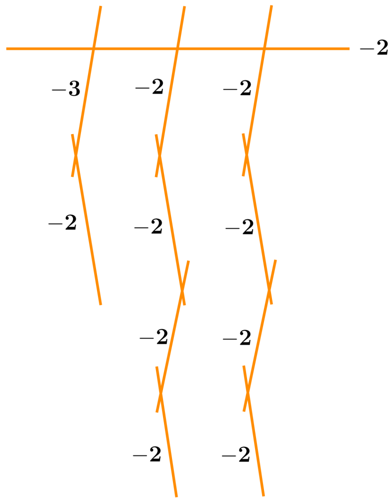

Figure 20: Order 5 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{2}{5}, \frac{4}{5}, \frac{4}{5}\right)$

Theorem 15.

$$\begin{aligned} e(X(2,5,3,3)) &= 66, \qquad c_1^2(X(2,5,3,3)) = 6, \\ \sigma(X(2,5,3,3)) &= -42, \qquad \chi(X(2,5,3,3)) = 6. \end{aligned}$$

$X(2,5,3,3)$, which is $Z(2)$ in **?** *(see also Akhmedov & Monden (2016)), and the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,3)$ is*

$$(\tau_5)^5 = 1.$$

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 11 \cdot 2 - 10 = 12.$$

Hence, by Lemma 3

$$e\left(X(2,5,3,3)\right)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 5 \cdot 12 + (2-5)\,(2-4) = 66.$$

There are 5 singular fibers, and each has monodromy given by τ_5.

Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,3)$ is $(\tau_5)^5 = 1$, by Lemma 2.

Recall that $\tau_2 = C_1C_2C_3C_4C_5^2C_4C_3C_2C_1$ and $\tau_5 = \tau_2C_1C_2C_3C_4$ by Lemma 2. τ_2 is represented as a product of 10 right-handed Dehn twists. So, τ_5 can be represented as a product of 14 right-handed Dehn twists. Thus, there are total $14 \cdot 5 = 70$ singular fibers.

By Lemma 2, we get

$$\sigma\left(X(2,5,3,3)\right) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 70 = -42.$$

$c_1^2\left(X(2,5,3,3)\right) = 6$ and $\chi\left(X(2,5,3,3)\right) = 6$, which follows from the formulas $c_1^2(X) = 2e(X) + 3\sigma(X)$ and $\chi(X) = \dfrac{e(X)+\sigma(X)}{4}$. $\square$

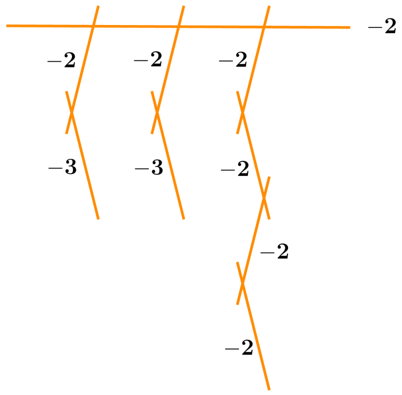

Figure 21: Order 5 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{3}{5}, \frac{3}{5}, \frac{4}{5}\right)$

6.4.4 Singular fiber of type $(\frac{3}{5}, \frac{3}{5}, \frac{4}{5})$

$$n_i = 5 \quad 1 \leq i \leq 3, \qquad q_1 = q_2 = 3, \quad q_3 = 4,$$
$$(Y)^2 = -\left(\frac{3}{5} + \frac{3}{5} + \frac{4}{5}\right) = -2,$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{5}{3} = [2,3], \\ \frac{n_3}{q_3} &= \frac{5}{4} = [2,2,2,2]. \end{aligned}$$

In this case, it can be seen from Figure 21 that there is no -1-sphere and the singular fibers correspond to type 44 in the table on pg. 359 in Ogg (1966) (see also Namikawa & Ueno (1973), type $IX-4$ on pg. 158).

Theorem 16.

$$\begin{aligned} e(X(2,5,3,4)) &= 56, \qquad c_1^2(X(2,5,3,4)) = 4, \\ \sigma(X(2,5,3,4)) &= -36, \qquad \chi(X(2,5,3,4)) = 5. \end{aligned}$$

$X(2,5,3,4)$ is diffeomorphic to the fiber sum of two copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus 2 fiber Σ_2, and the global monodromy of the genus two Lefschetz fibration on $X(2,5,3,4)$ is $(c_1c_2c_3c_4c_5^2)^{10} = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 9 \cdot 2 - 8 = 10.$$

Hence,

$$e(X(2,5,3,4)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 5 \cdot 10 + (-3) \cdot (2-4) = 56.$$

There are 5 singular fibers, and each has monodromy $(c_1c_2c_3c_4c_5^2)^2$. The later is determined using Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of $X(2,5,3,4)$ is

$$\left((c_1c_2c_3c_4c_5^2)^2\right)^5 = (c_1c_2c_3c_4c_5^2)^{10} = 1.$$

By Endo's signature formula for hyperelliptic Lefschetz fibrations,

$$\sigma(X(2,5,3,4)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 60 = -36.$$

Therefore, $c_1^2(X(2,5,3,4)) = 4$ and $\chi(X(2,5,3,4)) = 5$.

Thus, by the classification of genus two Lefschetz fibrations with non-separating singular fibers (Theorem 5.5 in Smith (1999)), $X(2,5,3,4)$ is the fiber sum of two copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus 2 fiber Σ_2.

□

6.5 $\mathbb{Z}_6$ Action On Σ_2

6.5.1 Singular fiber of type $(\frac{1}{6}, \frac{1}{6}, \frac{4}{6})$

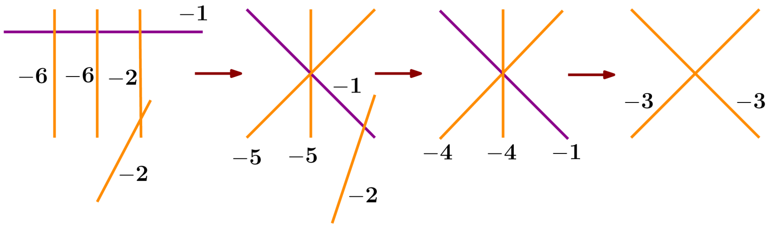

Figure 22: Order 6 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{1}{6}, \frac{1}{6}, \frac{4}{6}\right)$

$$n_i = 6 \quad 1 \leq i \leq 3, \qquad q_1 = q_2 = 1, \quad q_3 = 4,$$

$$(Y)^2 = -\left(\frac{1}{6} + \frac{1}{6} + \frac{4}{6}\right) = -1,$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{n_2}{q_2} = \frac{6}{1} = [6], \\ \frac{n_3}{q_3} &= \frac{6}{4} = \frac{3}{2} = [2,2]. \end{aligned}$$

In this case the singular fibers contain a central -1-sphere as illustrated in Figure 22 above. We blow down the central -1-sphere and get $X(2,6,3,1)$. The new fiber now corresponds to type 34 in the table on pg. 357 in Ogg (1966) (see also Namikawa & Ueno (1973), type V on pg. 156).

Theorem 17.

$$\begin{aligned} e(X(2,6,3,1)) &= 26, \qquad c_1^2(X(2,6,3,1)) = -2, \\ \sigma(X(2,6,3,1)) &= -18, \qquad \chi(X(2,6,3,1)) = 2. \end{aligned}$$

$X(2,6,3,1)$ is diffeomorphic to $K3\#2\overline{\mathbb{CP}}^2$, and the global monodromy of the genus two Lefschetz fibration on $X(2,6,3,1)$ is $(c_1c_2c_3c_4c_5)^6 = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 2 - 1 = 3.$$

Hence,

$$e(X(2,6,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 6 \cdot 3 + (-4) \cdot (2-4) = 26.$$

There are 6 singular fibers each has monodromy $c_1c_2c_3c_4c_5$. Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,6,3,1)$ is $(c_1c_2c_3c_4c_5)^6 = 1$.
By Endo's signature formula for hyperelliptic Lefschetz fibrations,

$$\sigma(X(2,6,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 30 = -18$$

Therefore, $c_1^2(X(2,6,3,1)) = -2$ and $\chi(X(2,6,3,1)) = 2$.
Hence, by Theorem 5.5 in Smith (1999), we conclude that $X(2,6,3,1)$ is $K3\#2\overline{\mathbb{CP}}^2$. □

6.5.2 singular fiber of type $(\frac{2}{6}, \frac{5}{6}, \frac{5}{6})$

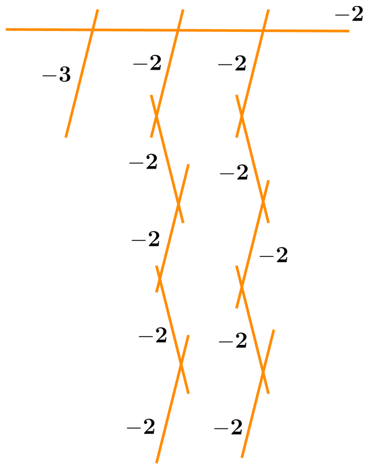

Figure 23: Order 6 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{2}{6}, \frac{5}{6}, \frac{5}{6}\right)$

$$n_i = 6 \quad 1 \le i \le 3, \qquad q_1 = 2, \quad q_2 = q_3 = 5,$$
$$(Y)^2 = -\left(\frac{2}{6} + \frac{5}{6} + \frac{5}{6}\right) = -2,$$

$$\frac{n_1}{q_1} = \frac{6}{2} = 3 = [3],$$
$$\frac{n_2}{q_2} = \frac{n_3}{q_3} = \frac{6}{5} = [2,2,2,2,2].$$

In this case, it can be seen in the Figure 23 above that there is no −1-sphere and the singular fibers correspond to type 19 in the table on pg. 358 in **?** (see also Namikawa & Ueno (1973), type V^* on pg. 156).

Theorem 18.

$$\begin{array}{rclr} e(X(2,6,3,2)) & = & 86, & c_1^2(X(2,6,3,2)) = 10, \\ \sigma(X(2,6,3,2)) & = & -54, & \chi(X(2,6,3,2)) = 8. \end{array}$$

$X(2,6,3,2)$ is diffeomorphic to the fiber sum of three copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus 2 fiber Σ_2, and the global monodromy of the genus two Lefschetz fibration on $X(2,6,3,2)$ is $(c_1c_2c_3c_4c_4c_5)^{18} = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 12 - 11 = 13.$$

Hence,

$$e(X(2,6,3,2)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 6 \cdot 13 + (-4) \cdot (2-4) = 86.$$

There are 6 singular fibers each has monodromy $(c_1c_2c_3c_4c_5)^3$. The later is determined using Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of $X(2,6,3,2)$ is

$$\left((c_1c_2c_3c_4c_5)^3\right)^6 = (c_1c_2c_3c_4c_5)^{18} = \left((c_1c_2c_3c_4c_5)^6\right)^3 = 1.$$

By Endo's signature formula for hyperelliptic Lefschetz fibrations,

$$\sigma(X(2,6,3,2)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 90 = -54.$$

Therefore, $c_1^2(X(2,6,3,2)) = 10$ and $\chi(X(2,6,3,2)) = 8$.

In conclusion, by the classification of genus two Lefschetz fibrations with non-separating singular fibers Smith (1999), we see that $X(2,6,3,2)$ is diffeomorphic to the fiber sum of three copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus 2 fiber Σ_2.

□

6.6 $\mathbb{Z}_8$ Action On Σ_2

6.6.1 Singular fiber of type $(\frac{1}{8}, (\frac{3}{8}, (\frac{4}{8})$

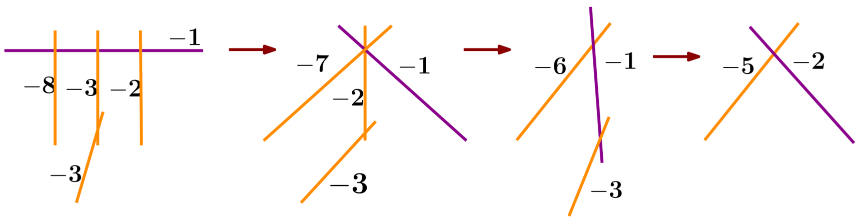

Figure 24: Order 8 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{1}{8}, \frac{3}{8}, \frac{4}{8}\right)$

$$n_i = 8 \quad 1 \le i \le 3, \qquad q_1 = 1, \quad q_2 = 3, \quad q_3 = 4,$$

$$(Y)^2 = -\left(\frac{1}{8} + \frac{3}{8} + \frac{4}{8}\right) = -1,$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{8}{1} = [8], \\
\frac{n_2}{q_2} &= \frac{8}{3} = [3,3], \\
\frac{n_3}{q_3} &= \frac{8}{4} = 2 = [2].
\end{aligned}$$

In this case, each singular fiber has a central -1-sphere (see Figure 24) and after 3 blow-down operations, the singular fibers correspond to type $VIII-1$ on pg. 156.

Theorem 19.

$$\begin{aligned}
e(X(2,8,3,1)) &= 36, & c_1^2(X(2,8,3,1)) = 0, \\
\sigma(X(2,8,3,1)) &= -24, & \chi_h(X(2,8,3,1)) = 3.
\end{aligned}$$

$X(2,8,3,1)$ is the elliptic surface $E(3)$, and the global monodromy of the genus two Lefschetz fibration on $X(2,8,3,1)$ is

$$(c_1c_2c_3c_4c_5^2c_4c_3c_2c_2)^4 = 1.$$

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 2 - 1 = 3.$$

Hence,

$$e(X(2,8,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 8 \cdot 3 + (-6) \cdot (2-4) = 36.$$

By applying Lemma 2, we compute

$$\sigma(X(2,8,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 40 = -24$$

Therefore, $c_1^2(X(2,8,3,1)) = 0$ and $\chi(X(2,8,3,1)) = 3$.

As a summary, we have a genus two Lefschetz fibration on $X(2,8,3,1)$ with global monodromy $(c_1c_2c_3c_4c_5^2c_4c_3c_2c_2)^4 = 1$ and the singular fibers contain only non-separating vanishing cycles. Thus, by Theorem 5.5 in Smith (1999), $X(2,8,3,1)$ is diffeomorphic to the elliptic surface $E(3)$. □

6.6.2 Singular fiber of type $(\frac{4}{8}, \frac{5}{8}, \frac{7}{8})$

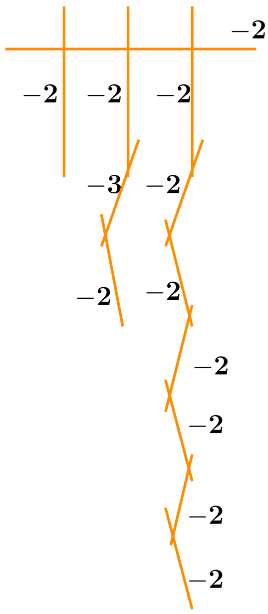

Figure 25: Order 8 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{4}{8}, \frac{5}{8}, \frac{7}{8}\right)$

$$n_i = 8 \quad 1 \leq i \leq 3, \qquad q_1 = 4, \quad q_2 = 5, \quad q_3 = 7,$$
$$(Y)^2 = -\left(\frac{4}{8} + \frac{5}{8} + \frac{7}{8}\right) = -2,$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{8}{4} = 2 = [2], \\ \frac{n_2}{q_2} &= \frac{8}{5} = [2, 3, 2], \\ \frac{n_3}{q_3} &= \frac{8}{7} = [2, 2, 2, 2, 2, 2, 2]. \end{aligned}$$

In this case, the singular fibers correspond to type 22 in the table on pg. 358 in Ogg (1966) (see also Namikawa & Ueno (1973), type VII^* on pg. 156).

Theorem 20.

$$\begin{aligned} e(X(2,8,3,2)) &= 116, \qquad c_1^2(X(2,8,3,2)) = 16, \\ \sigma(X(2,8,3,2)) &= -72, \qquad \chi(X(2,8,3,2)) = 11. \end{aligned}$$

$X(2,8,3,2)$ is diffeomorphic to the fiber sum of four copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus $g = 2$ fiber, and the global monodromy of the genus two Lefschetz fibration on $X(2,8,3,2)$ is $(c_1c_2c_3c_4c_5)^{24} = 1$.

Proof. Note that each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 12 - 11 = 13.$$

We compute

$$e(X(2,8,3,2)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 8 \cdot 13 + (-6) \cdot (2-4) = 116.$$

There are 8 singular fibers each has monodromy $(c_1c_2c_3c_4c_5)^3$, the later follows from Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of $X(2,8,3,2)$ is

$$\left((c_1c_2c_3c_4c_5)^3\right)^8 = (c_1c_2c_3c_4c_5)^{24} = \left((c_1c_2c_3c_4c_5)^6\right)^4 = 1.$$

By applying Endo's signature formula for hyperelliptic Lefschetz fibrations, we compute

$$\sigma(X(2,8,3,2)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 120 = -72$$

Consequently, we have $c_1^2(X(2,8,3,2)) = 16$ and $\chi(X(2,8,3,2)) = 11$.

Hence, using the classification of genus two Lefschetz fibrations with non-separating singular fibers Smith (1999), we conclude that $X(2,8,3,2)$ is diffeomorphic to the fiber sum of four copies of $K3\#2\overline{\mathbb{CP}}^2$ along the genus $g = 2$ fiber.

□

6.7 $\mathbb{Z}_{10}$ Action On Σ_2

6.7.1 Singular fiber of type $(\frac{1}{10}, \frac{4}{10}, \frac{5}{10})$

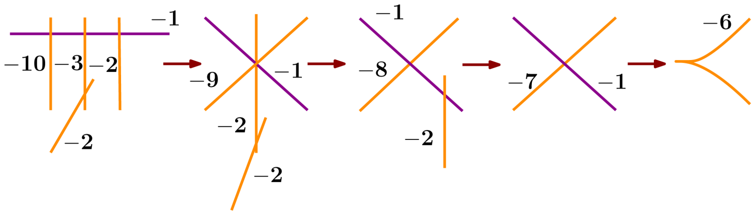

Figure 26: Order 10 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{1}{10}, \frac{4}{10}, \frac{5}{10}\right)$

$$n_i = 10 \quad 1 \le i \le 3, \qquad q_1 = 1, \quad q_2 = 4, \quad q_3 = 5,$$
$$(Y)^2 = -\left(\frac{1}{10} + \frac{4}{10} + \frac{5}{10}\right) = -1,$$

$$\frac{n_1}{q_1} = \frac{10}{1} = [10],$$
$$\frac{n_2}{q_2} = \frac{10}{4} = \frac{5}{2} = [3,2],$$
$$\frac{n_3}{q_3} = \frac{10}{5} = 2 = [2].$$

Theorem 21.

$$e(X(2,10,3,1)) = 36, \qquad c_1^2(X(2,10,3,1)) = 0,$$
$$\sigma(X(2,10,3,1)) = -24, \qquad \chi(X(2,10,3,1)) = 3.$$

$X(2,10,3,1)$ is the Horikawa surface, and the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,1)$ is $(c_1c_2c_3c_4)^{10} = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 1 - 0 = 2.$$

Hence,

$$e(X(2,10,3,1)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 10 \cdot 2 + (2-10) \cdot (2-4) = 36.$$

There are 10 singular fibers each has monodromy given by $c_1c_2c_3c_4$, which follows from Ishizaka's classification of the periodic monodromies given in Ishizaka (2007). Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,1)$ is $(c_1c_2c_3c_4)^{10} = 1$.
Applying, Endo's signature formula for hyperelliptic Lefschetz fibrations, we have

$$\sigma(X(2,10,3,1)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1 \right) s_h = -\frac{3}{5} \cdot 40 = -24.$$

Consequently, we have $c_1^2(X(2,10,3,1)) = 0$ and $\chi(X(2,10,3,1)) = 3$.
It follows by Theorem 5.5 in Smith (1999) that $X(2,10,3,1)$ is diffeomorphic to the Horikawa's surface.

□

6.7.2 Singular fiber of type $(\frac{5}{10}, \frac{6}{10}, \frac{9}{10})$

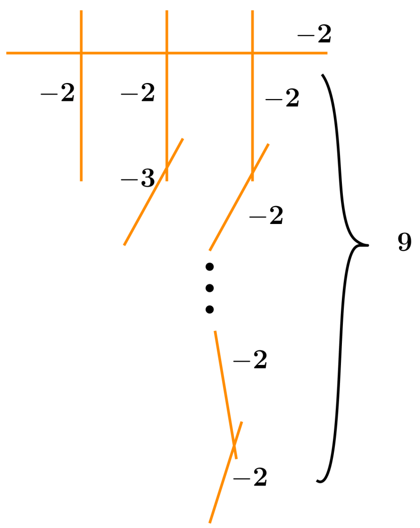

Figure 27: Order 10 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{5}{10}, \frac{6}{10}, \frac{9}{10}\right)$

$$n_i = 10 \quad 1 \leq i \leq 3, \qquad q_1 = 5, \quad q_2 = 6, \quad q_3 = 9,$$

$$(Y)^2 = -\left(\frac{5}{10} + \frac{6}{10} + \frac{9}{10}\right) = -2,$$

$$\begin{aligned}
\frac{n_1}{q_1} &= \frac{10}{5} = 2 = [2], \\
\frac{n_2}{q_2} &= \frac{10}{6} = \frac{5}{3} = [2,3], \\
\frac{n_3}{q_3} &= \frac{10}{9} = [2,2,2,2,2,2,2,2,2].
\end{aligned}$$

In this case, the singular fibers correspond to type 20 in the table on pg. 358 in Ogg (1966) (see also Namikawa & Ueno (1973), type $VIII-4$ on pg. 157).

Theorem 22.

$$\begin{aligned} e(X(2,10,3,2)) &= 156, & c_1^2(X(2,10,3,2)) = 24, \\ \sigma(X(2,10,3,2)) &= -96, & \chi(X(2,10,3,2)) = 15. \end{aligned}$$

$X(2,10,3,2)$ is diffeomorphic to the fiber sum of four copies of Horikawa's surface, and the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,2)$ is $(c_1c_2c_3c_4)^{40} = 1$.

Proof. Each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 13 - 12 = 14.$$

Hence,

$$e(X(2,10,3,2)) = n \cdot e(F_s) + (2-n)\cdot(2-2g) = 10 \cdot 14 + (-8)\cdot(2-4) = 156.$$

There are 10 singular fibers each has monodromy $(c_1c_2c_3c_4)^4$.
Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,2)$ is

$$\left((c_1c_2c_3c_4)^4\right)^{10} = (c_1c_2c_3c_4)^{40} = \left((c_1c_2c_3c_4)^{10}\right)^4 = 1.$$

Using Lemma 2, we compute

$$\sigma(X(2,10,3,2)) = -\frac{g+1}{2g+1}\cdot n - \sum_{h=1}^{[g/2]}\left(\frac{4h\,(g-h)}{2g+1} - 1\right)s_h = -\frac{3}{5}\cdot 160 = -96.$$

Consequently, we have $c_1^2(X(2,10,3,2)) = 24$ and $\chi(X(2,10,3,2)) = 15$.
Again, using the classification of genus two Lefschetz fibrations with non-separating singular fibers, which is due to Chakiris, (see Theorem 5.5 in Smith (1999)), we see that $X(2,10,3,2)$ is diffeomorphic to the fiber sum of four copies of Horikawa's surface.

□

6.7.3 Singular fiber of type $(\frac{5}{10}, \frac{7}{10}, \frac{8}{10})$

$$n_i = 10 \quad 1 \le i \le 3, \qquad q_1 = 5, \quad q_2 = 7, \quad q_3 = 8,$$
$$(Y)^2 = -\left(\frac{5}{10} + \frac{7}{10} + \frac{8}{10}\right) = -2,$$

$$\begin{aligned} \frac{n_1}{q_1} &= \frac{10}{5} = 2 = [2], \\ \frac{n_2}{q_2} &= \frac{10}{7} = \frac{10}{7} = [2,2,4], \\ \frac{n_3}{q_3} &= \frac{10}{8} = \frac{5}{4} = [2,2,2,2]. \end{aligned}$$

In this case, the singular fibers correspond to type 7 in the table on pg. 357 in Ogg (1966) (see also Namikawa & Ueno (1973), type $VIII-2$ on pg. 157).

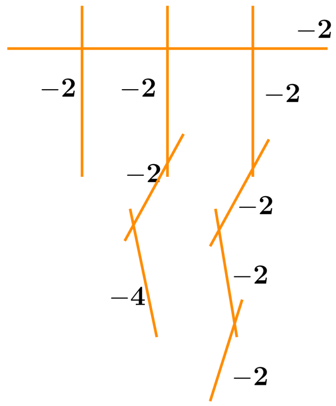

Figure 28: Order 10 action on Σ_2 with 3 fixed points, singular fiber of type $\left(\frac{5}{10}, \frac{7}{10}, \frac{8}{10}\right)$

Theorem 23.

$$\begin{aligned} e(X(2,10,3,3)) &= 116, \qquad c_1^2(X2,10,3,3)) = 16, \\ \sigma(X(2,10,3,3)) &= -72, \qquad \chi(X(2,10,3,3)) = 11. \end{aligned}$$

$X(2,10,3,3)$ is the fiber sum of three copies of Horikawa's surface, and the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,3)$ is $(c_1c_2c_3c_4)^{30} = 1$.

Proof. Notice that each singular fiber has Euler characteristic

$$e(F_s) = 2 \cdot 9 - 8 = 10.$$

We compute

$$e(X(2,10,3,3)) = n \cdot e(F_s) + (2-n) \cdot (2-2g) = 10 \cdot 10 + (-8) \cdot (2-4) = 116.$$

There are 10 singular fibers and each has monodromy $(c_1c_2c_3c_4)^3$. The later follows from Ishizaka's classification of the periodic monodromies given in Ishizaka (2007).

Thus, the global monodromy of the genus two Lefschetz fibration on $X(2,10,3,3)$ is

$$\left((c_1c_2c_3c_4)^3\right)^{10} = (c_1c_2c_3c_4)^{30} = \left((c_1c_2c_3c_4)^{10}\right)^3 = 1.$$

By applying Lemma 2, we compute

$$\sigma(X(2,10,3,3)) = -\frac{g+1}{2g+1} \cdot n - \sum_{h=1}^{[g/2]} \left(\frac{4h\,(g-h)}{2g+1} - 1\right) s_h = -\frac{3}{5} \cdot 120 = -72.$$

Consequently, we have $c_1^2(X(2,10,3,3)) = 16$ and $\chi(X(2,10,3,3)) = 11$.

Thus, by the classification of genus two Lefschetz fibrations with non-separating singular fibers (Theorem 5.5 in Smith (1999)), $X(2,10,3,3)$ is the fiber sum of three copies of Horikawa's surface.

□

Remark. *Using Theorems 12, 14, 19, 23, one can see that the corresponding Lefschetz fibrations contain some plumbed negative definite configurations of spheres that can be rationally blown down. For example, these Lefschetz fibrations contain the rational blowdown plumbings C_2 and C_3, which can be found in the Figures 17, 19, 24, 28. One can construct new Lefschetz fibration by applying the rational blowdown operation to these singular fibers, which corresponds to a daisy relation in the mapping class group. We study family of such Lefschetz fibrations and and discuss further applications in Akhmedov & Nur Saglam Kadriye (2018).*

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